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DEVELOPMENT OF THE BIFURCATION CONDITION FOR A THICK ELASTIC PLATE

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Abstract—A consistent six-term asymptotic series is derived for the critical value of the deformation parameter as a function of aspect ratio (thickness: height), in the limit as this ratio becomes unbounded, for a plate of general incompressible isotropic material. Values for both flexural and barreling modes are obtained in the case where the loading is a thrust. The theoretically important class of Mooney–Rivlin materials is treated as a special case.

INTRODUCTION

The condition describing the critical deformation at which a thick plate of incompressible elastic material may undergo a bifurcation might now be regarded as a classical result in the literature on finite elasticity, its having been derived in various contexts by a number of workers. An account of some of this work is given in Refs [1–3]. Briefly, the bifurcation condition is an equation relating a deformation parameter, λ, and a geometric factor, η, proportional to the plate's aspect ratio (thickness: height), the solution of which *is* the critical value of λ (see eqn (4)).

Other workers, including Biot[4], Nowinski[5] and Usmani and Beatty[6], related the appearance of wrinkles on the surface of a "half-space" of Mooney–Rivlin (or neo-Hookean) material to bifurcations of a thick plate, in the limiting case $\eta \to \infty$. The purpose of this paper is to develop a consistent basis for carrying forward the analysis pertaining to the connection between the thick plate and its idealized limit.

Recently, Sawyers and Rivlin[7] employed Koiter's theory[8] and developed the analysis required to decide the stability (or lack thereof) of the underlying critical states of deformation for a plate of arbitrary aspect ratio the material behavior of which is modeled by a completely general strain-energy function. The resulting formula is complicated and acquires meaning only through numerical calculations. Towards this end, an analysis was carried out for the limiting case of a thin plate, $\eta \to 0$.

A preliminary result, on which this analysis was based, was obtained much earlier[9], and came directly from the bifurcation equation itself. This amounted to establishing a consistent series approximation for λ (near unity) in terms of η (near zero). A striking feature of this result, for the thin plate, is the essential insensitivity of λ to details of the strain-energy function, at least up to all relevant powers of η required for the stability analysis in Ref. [7].

In contrast to this, an immediate inference drawn from Refs [2, 3] is that, for large η, the critical value of λ depends strongly on the constitutive assumption. Specifically, as $\eta \to \infty$, the critical value of λ tends to a limiting number, $\bar{\lambda}$, which can be determined only by finding the root of a certain equation; and this procedure cannot even begin until the form of the strain-energy function is stated explicitly (see eqn (9)). Nevertheless, it is possible to carry the analysis forward, much in the spirit of Ref. [9], leaving the root-finding problem as the only remaining task.

To keep the work here as uncomplicated as possible, while retaining a reasonable degree of generality, we shall restrict attention to the class of materials for which flexural and barreling bifurcations cannot occur simultaneously (i.e. at a single value of λ) for any finite value of η. This class includes the Mooney–Rivlin (and neo-Hookean) material for which the following result is known (see Refs [1, 2, 4–6]): $\lambda \to 3.383$ as $\eta \to \infty$.

BACKGROUND

The body under consideration is a rectangular plate of incompressible material the initial height, thickness and width of which are $2l_1$, $2l_2$, $2l_3$, respectively, with $l_3 \gg l_1$, l_2. Uniformly distributed normal forces are applied to the vertical end faces, thereby inducing a stretch in the width direction, with extension ratio λ_3. No further deformation occurs in the width direction. The top and bottom faces are subjected to equal and opposite uniformly distributed normal thrust forces, while the major vertical faces remain free of tractions. Under these conditions the plate tends to shorten in the vertical direction and its thickness tends to increase, the extension ratios in these directions being λ_1, λ_2, respectively. A single parameter suffices to describe this underlying state of deformation, which we take to be $\lambda = \lambda_2/\lambda_1$. As the thrust is increased, λ increases from unity and, at a certain critical value of λ, a bifurcation in the form of a small superposed deformation in the plane normal to the width direction becomes possible.

The strain-energy, W, per unit volume, is expressible as a function of I_1 and I_2, defined in terms of λ and λ_3 by

$$I_1 = (\lambda + 1/\lambda)/\lambda_3 + \lambda_3^2, \qquad I_2 = \lambda_3(\lambda + 1/\lambda) + 1/\lambda_3^2. \tag{1}$$

It has been shown that the bifurcation equation depends on W only through a single material parameter which may be taken as[2, 3, 9]

$$A = (2/\lambda_3)(\sqrt{\lambda} + 1/\sqrt{\lambda})^2 S/T \equiv A(\lambda) \tag{2}$$

where†

$$T = W_1 + \lambda_3^2 W_2, \qquad S = W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}. \tag{3}$$

We shall here restrict attention to materials for which $A > -1$, i.e. to materials for which flexural and barreling modes cannot appear simultaneously. Accordingly, the bifurcation equation may be written as[2]

$$\sinh Q\eta/\sinh R\eta = \nu(Q/R)\{R^2 + \lambda(\lambda+1)^2\}/\{Q^2 - \lambda(\lambda-1)^2\} \tag{4}$$

where

$$Q = (\lambda-1)\{A + (\lambda+1)^2/(\lambda-1)^2\}^{1/2}, \qquad R = (\lambda-1)(A+1)^{1/2} \tag{5}$$

and where

$$\eta = n\pi l_2/(2l_1), \qquad n = 1, 2, 3, \ldots \tag{6}$$

n being the number of half-wavelengths along the vertical direction in the mode considered. The numerical parameter ν is 1 or -1 accordingly as the bifurcation is of flexural or barreling type.

In Ref. [2] it was proven that the demarcation between flexural and barreling regimes is the curve in the A–λ plane defined by the vanishing of the denominator in the right-hand member of eqn (4), namely

$$Q^2 = \lambda(\lambda-1)^2 \tag{7}$$

or, with eqn $(5)_1$

† We use the notation $W_1 = \partial W/\partial I_1$, $W_{12} = \partial^2 W/\partial I_1\,\partial I_2$, etc.

$$A = \lambda - (\lambda+1)^2/(\lambda-1)^2 \equiv \bar{A}(\lambda). \tag{8}$$

This separatrix is plotted as Fig. 3 of Ref. [2]. Now for a specified material and for a specified value of λ_3, the relevant A function can be calculated, as a function of λ, using eqns (1) and (2), and the result compared with $\bar{A}(\lambda)$ in eqn (8). If $A(\lambda) > \bar{A}(\lambda)$, only flexural modes are possible. The case of interest here, however, is where the material function $A(\lambda)$ intersects the separatrix so that, at least for some values of λ, we have $A(\lambda) < \bar{A}(\lambda)$. Then barreling modes may appear. We shall assume that this is the case and denote by $\bar{\lambda}$ the (smallest) value of λ for which (cf. eqn (8))

$$A(\bar{\lambda}) = \bar{A}(\bar{\lambda}) = \bar{\lambda} - (\bar{\lambda}+1)^2/(\bar{\lambda}-1)^2 \equiv \bar{A}. \tag{9}$$

Of necessity, as shown in Ref. [2]

$$\bar{\lambda} > 3 \qquad \text{if } \bar{A} > -1. \tag{10}$$

Further, from Ref. [2] it is known that $\eta \to \infty$ as $\lambda \to \bar{\lambda}$, and vice versa, and it is the investigation of the solutions of eqn (4) in this limit that is the object of this paper. We remark here that flexural and barreling cases can be handled simultaneously, without undue complication, through the explicit appearance of ν in eqn (4). Previous results suggest, however, that flexural deformations may be inherently unstable in this "thick-plate" limit[10, 11].†

ANALYSIS

We begin by expanding quantities that appear in the right-hand member of eqn (4) in the Taylor series about $\bar{\lambda}$. To facilitate this, from eqns (5) we verify the general result, $Q^2 = 4\lambda + R^2$, valid for arbitrary values of λ. Then

$$\mathrm{d}Q^2/\mathrm{d}\lambda = 4 + \mathrm{d}R^2/\mathrm{d}\lambda, \qquad \mathrm{d}^jQ^2/\mathrm{d}\lambda^j = \mathrm{d}^jR^2/\mathrm{d}\lambda^j, \qquad j = 2, 3, \ldots \tag{11}$$

From eqns (5) and (9) follow

$$\bar{Q} = \sqrt{\bar{\lambda}}(\bar{\lambda}-1), \qquad \bar{R} = \sqrt{\bar{\lambda}}\{(\bar{\lambda}+1)(\bar{\lambda}-3)\}^{1/2} \tag{12}$$

where we use the notation $\bar{Q} = Q(\bar{\lambda})$, etc. The eqns (11) and (12) yield

$$\begin{aligned} R^2 &= \bar{R}^2\{1 + r_1 d + \tfrac{1}{2}r_2 d^2 + \tfrac{1}{6}r_3 d^3 + O(d^4)\} \\ Q^2 &= \bar{Q}^2\{1 + (q_1 + 4/\bar{Q}^2)d + \tfrac{1}{2}q_2 d^2 + \tfrac{1}{6}q_3 d^3 + \tfrac{1}{24}q_4 d^4 + O(d^5)\} \end{aligned} \tag{13}$$

where

$$d = \lambda - \bar{\lambda} \tag{14}$$

and

$$r_j = \{(\mathrm{d}^jR^2/\mathrm{d}\lambda^j)|_{\bar{\lambda}}\}/\bar{R}^2, \qquad q_j = \bar{R}^2 r_j/\bar{Q}^2, \qquad j = 1, 2, \ldots \tag{15}$$

The application of the Maclaurin series $(1+x)^{1/2} = 1 + x/2 - x^2/8 + x^3/16 + \ldots$ to eqns (13) yields

† Specifically, no flexural mode is stable for the neo-Hookean material if $\eta > 0.32$, corresponding to $\lambda > 1.07$.

$$R = \bar{R}\{1+R_1 d+R_2 d^2+R_3 d^3+O(d^4)\}$$
$$Q = \bar{Q}\{1+Q_1 d+Q_2 d^2+Q_3 d^3+O(d^4)\} \tag{16}$$

where

$$R_1 = r_1/2, \qquad R_2 = (r_2 - r_1^2/2)/4$$
$$R_3 = (r_3 - 3r_1 r_2/2 + 3r_1^3/4)/12$$
$$Q_1 = (q_1 + 4/\bar{Q}^2)/2, \qquad Q_2 = \{q_2 - (q_1 + 4/\bar{Q}^2)^2/2\}/4 \tag{17}$$
$$Q_3 = \{q_3 - 3q_2(q_1 + 4/\bar{Q}^2)/2 + 3(q_1 + 4/\bar{Q}^2)^3/4\}/12.$$

Noting that $\lambda = \bar{\lambda}+d$, $(\lambda+1) = \bar{\lambda}+1+d$, etc., we use eqns (13), with eqns (12), to obtain

$$R^2+\lambda(\lambda+1)^2 = 2\bar{\lambda}(\bar{\lambda}^2-1)\{1+N_1 d+N_2 d^2+N_3 d^3+O(d^4)\}$$
$$Q^2-\lambda(\lambda-1)^2 = -(\lambda-1)^2 D_1 d\{1-D_2 d-D_3 d^2-D_4 d^3+O(d^4)\} \tag{18}$$

where

$$N_1 = \{\bar{R}^2 r_1 + (\bar{\lambda}+1)(3\bar{\lambda}+1)\}/\{2\bar{\lambda}(\bar{\lambda}^2-1)\}$$
$$N_2 = \{\tfrac{1}{2}\bar{R}^2 r_2 + 3\bar{\lambda}+2\}/\{2\bar{\lambda}(\bar{\lambda}^2-1)\}$$
$$N_3 = \{\tfrac{1}{6}\bar{R}^2 r_3 + 1\}/\{2\bar{\lambda}(\bar{\lambda}^2-1)\}$$
$$D_1 = \{(\bar{\lambda}-1)(3\bar{\lambda}-1) - \bar{Q}^2 q_1 - 4\}/(\bar{\lambda}-1)^2 \tag{19}$$
$$D_2 = \{\tfrac{1}{2}\bar{Q}^2 q_2 - 3\bar{\lambda}+2\}/\{(\bar{\lambda}-1)^2 D_1\}$$
$$D_3 = \{\tfrac{1}{6}\bar{Q}^2 q_3 - 1\}/\{(\bar{\lambda}-1)^2 D_1\}$$
$$D_4 = \tfrac{1}{24}\bar{Q}^2 q_4/\{(\bar{\lambda}-1)^2 D_1\}.$$

The quantity D_1 has special geometrical significance. To see this, we use eqn (9) to calculate the slope of the separatrix at $\bar{\lambda}$, namely

$$\bar{A}'(\bar{\lambda}) = (\bar{\lambda}^3 - 3\bar{\lambda}^2 + 7\bar{\lambda} + 3)/(\bar{\lambda}-1)^3 \tag{20}$$

and then employ eqns (15) (with $j = 1$) and (12) in eqn $(19)_4$. The result is found to be

$$D_1 = \bar{A}'(\bar{\lambda}) - A'(\bar{\lambda}). \tag{21}$$

Now in view of eqn (9), a sufficient condition that the material curve $A(\lambda)$ vs λ actually cross the separatrix at the point $(\bar{\lambda}, \bar{A})$ is $\bar{A}'(\bar{\lambda}) > A'(\bar{\lambda})$, which we shall assume. Hence, we assume $D_1 > 0$.

The substitution from eqns (16) and (18) into the right-hand member of eqn (4), with eqns (12), yields

$$\nu \frac{Q}{R}\frac{R^2+\lambda(\lambda+1)^2}{Q^2-\lambda(\lambda-1)^2} = -\bar{\lambda}\nu(d_0/d)\{1+\delta_1 d+\delta_2 d^2+\delta_3 d^3+O(d^4)\} \tag{22}$$

where

$$d_0 = 2\{(\bar{\lambda}+1)/(\bar{\lambda}-3)\}^{1/2}/D_1 \tag{23}$$

and

$$\begin{aligned}\delta_1 &= N_1+D_2+Q_1-R_1\\ \delta_2 &= N_2+D_3+Q_2-R_2+(N_1+D_2)(D_2+Q_1-R_1)-R_1(Q_1-R_1)\\ \delta_3 &= N_3+D_4+Q_3-R_3+(N_2+D_3+N_1D_2+D_2^2)(D_2+Q_1-R_1)\\ &\quad +(Q_2-Q_1R_1+R_1^2-R_2)(N_1+D_2-R_1)+D_3(N_1+D_2)-R_2(Q_1-R_1).\end{aligned} \tag{24}$$

Expansion of the left-hand member of eqn (4) yields

$$\frac{\sinh Q\eta}{\sinh R\eta} = e^{(Q-R)\eta}\{1+e^{-2R\eta}-e^{-2Q\eta}+e^{-4R\eta}-e^{-2(Q+R)\eta}+e^{-6R\eta}-e^{-2(Q+2R)\eta}+\langle -8R\eta\rangle\} \tag{25}$$

where we employ the notation $\langle p\rangle = O(e^p)$. To assist in deciding the relative orders of certain quantities that will arise from further expansion of the terms in eqn (25), we note from eqns (12) that

$$0 > -(\bar{Q}-\bar{R}) > -2(\bar{Q}-\bar{R}) \qquad \text{if } \bar{\lambda} > 3 \tag{26}$$

and further, that

$$-2(\bar{Q}-\bar{R}) > -2\bar{R} > -3(\bar{Q}-\bar{R}) > -(\bar{Q}+\bar{R}) > -4(\bar{Q}-\bar{R}) > -2\bar{Q} \qquad \text{if } 1+4/\sqrt{3} < \bar{\lambda} < 3.5. \tag{27}$$

The use of eqn (9) shows that restrictions (27) apply to any material for which

$$-(3-4/\sqrt{3})/4 < A(\bar{\lambda}) < 13/50. \tag{28}$$

The Mooney–Rivlin material ($A \equiv 0$) is seen to be in this class.

From eqns (16)

$$Q-R = \Delta\{1+E_1d+E_2d^2+E_3d^3+O(d^4)\} \tag{29}$$

where

$$\Delta = \bar{Q}-\bar{R}, \qquad E_j = (\bar{Q}Q_j-\bar{R}R_j)/\Delta, \qquad j=1,2,3. \tag{30}$$

We now bear in mind inequalities (26) and (27) and conclude that, in the limit $\eta \to \infty$, the dominant term in eqn (25) is exp $(\Delta\eta)$. The first approximation to d follows upon equating this to the dominant term in eqn (22). Thus, $d \approx -\bar{\lambda}vd_0 \exp(-\Delta\eta)$ as $\eta \to \infty$. We introduce the notation

$$D_0 = -\bar{\lambda}vd_0 \tag{31}$$

refer again to inequalities (26) and (27), and conclude that a consistent asymptotic approximation to d should be of the form

$$d = D_0\, e^{-\Delta\eta}\{1+a_1\, e^{-\Delta\eta}+a_2\, e^{-2\Delta\eta}+b\, e^{-2R\eta}+a_3\, e^{-3\Delta\eta}+\langle -(\bar{Q}+\bar{R})\eta\rangle\} \tag{32}$$

where the a's and b remain to be determined.

From eqn (32) we find

$$\begin{aligned} d^2 &= D_0^2\, e^{-2\Delta\eta}\{1+2a_1\, e^{-\Delta\eta}+\langle -2\Delta\eta\rangle\}\\ d^3 &= D_0^3\, e^{-3\Delta\eta}\{1+\langle -\Delta\eta\rangle\}.\end{aligned} \tag{33}$$

The expansion of terms of the form exp $\{\alpha \exp(-\beta\eta)\}$, with $\beta > 0$, will be needed. For large η, we note that

$$\exp(\alpha\, e^{-\beta\eta}) = 1+\alpha\, e^{-\beta\eta}+\tfrac{1}{2}\alpha^2\, e^{-\beta\eta}+\tfrac{1}{6}\alpha^3\, e^{-3\beta\eta}+\langle -4\beta\eta\rangle. \tag{34}$$

Then, upon employing eqns (32) and (34) in eqns (16) and (29), and substituting the results into eqn (25), we obtain

$$\frac{\sinh Q\eta}{\sinh R\eta} = e^{\Delta\eta}\{1+C_1\, e^{-\Delta\eta}+C_2\, e^{-2\Delta\eta}+e^{-2\bar{R}\eta}+C_3\, e^{-3\Delta\eta}+\langle -(\bar{Q}+\bar{R})\eta\rangle\} \tag{35}$$

where

$$\begin{aligned} C_1 &= \Delta\eta D_0 E_1 \\ C_2 &= \Delta\eta D_0\{E_1 a_1+D_0E_2+\tfrac{1}{2}\Delta\eta D_0E_1^2\} \\ C_3 &= \Delta\eta D_0\{E_1a_2+2D_0E_2a_1+D_0^2E_3+\Delta\eta D_0E_1(E_1a_1+D_0E_2)+\tfrac{1}{6}\Delta^2\eta^2D_0^2E_1^3\}. \end{aligned} \tag{36}$$

Likewise, the use of eqns (32) and (33) in eqn (22), followed by a comparison of the result with that of eqn (35), leads to the conclusion that

$$\begin{aligned} C_1 &= D_0\delta_1-a_1, \qquad C_2 = a_1^2+D_0^2\delta_2-a_2, \\ C_3 &= D_0^2\delta_2a_1+D_0^3\delta_3-a_1^3+2a_1a_2-a_3 \end{aligned} \tag{37}$$

and

$$b = -1. \tag{38}$$

RESULTS

The main result of this paper is the development of an asymptotic expression for the critical value of λ, valid for large values of η, these quantities being related by eqn (4). From eqns (14) and (32), with eqns (31) and (38) we conclude that

$$\lambda = \bar{\lambda}\{1-\nu d_0[e^{-\Delta\eta}+a_1\, e^{-2\Delta\eta}+a_2\, e^{-3\Delta\eta}-e^{-(\bar{Q}+\bar{R})\eta}+a_3\, e^{-4\Delta\eta}+\langle -2\bar{Q}\eta\rangle]\} \tag{39}$$

where $\bar{\lambda}$ is the smallest value of λ that satisfies eqn (9), d_0 is given by eqn (23), $\Delta = \bar{Q}-\bar{R}$, and ν is assigned the value 1 or -1 accordingly as flexural or barreling modes are considered. Explicit expressions for the a's follow directly from eqns (36) and (37) upon equating the corresponding C's. Thus, with eqn (31)

$$\begin{aligned} a_1 &= -\bar{\lambda}\nu d_0\{\delta_1-\Delta\eta E_1\} \\ a_2 &= \bar{\lambda}^2 d_0^2\{\delta_1^2+\delta_2-\Delta\eta(3\delta_1E_1+E_2)+\tfrac{3}{2}\Delta^2\eta^2E_1^2\} \\ a_3 &= -\bar{\lambda}^3\nu d_0^3\{\delta_1^3+3\delta_1\delta_2+\delta_3-\Delta\eta(4\delta_1E_2+4\delta_2E_1+6\delta_1^2E_1+E_3) \\ &\qquad +4\Delta^2\eta^2E_1(E_2+2\delta_1E_1)-\tfrac{8}{3}\Delta^3\eta^3E_1^3\} \end{aligned} \tag{40}$$

where the δ's and E's are defined in eqns (24) and (30).

The terms that appear in eqn (39) are written in strictly decreasing order of exponential magnitudes, provided the material function $A(\lambda)$ satisfies restrictions (28). If it does not, then certain inequalities in the chain (27) would be violated, and an alternate set of terms in eqn (32) would be required. Details pertaining to such cases shall not be addressed here.

Although the determination of $\bar{\lambda}$ involves only knowledge of the values of $A(\lambda)$, through eqn (9), the quantities a_i in eqns (40) depend on various derivatives of A, evaluated at $\lambda = \bar{\lambda}$.

In particular, it is found that d_0 depends on $A'(\bar{\lambda})$, E_j depends on $A^{(j)}(\bar{\lambda})$, and δ_j depends on $A^{(j+1)}(\bar{\lambda})$. To see how these quantities are related to derivatives of the strain-energy function itself, we employ eqns (1)–(3) and obtain, for an arbitrary value of λ

$$T' = \lambda_3^{-1}(1-1/\lambda^2)S, \qquad S' = \lambda_3^{-1}(1-1/\lambda^2)U, \qquad S/T = \tfrac{1}{2}\lambda_3\lambda(\lambda+1)^{-2}A \tag{41}$$

where

$$U = W_{111}+3\lambda_3^2W_{112}+3\lambda_3^4W_{122}+\lambda_3^6W_{222}. \tag{42}$$

Thus, in general

$$A' = \frac{(\lambda-1)}{\lambda(\lambda+1)}A\left(1-\frac{1}{2}A\right)+\frac{2}{\lambda_3^2}(\lambda-1)(1+1/\lambda)^3U/T \tag{43}$$

and, with eqn (9), we find

$$A'(\bar{\lambda}) = \tfrac{1}{2}(\bar{\lambda}-1)\{\bar{\lambda}(\bar{\lambda}+1)\}^{-1}\bar{A}(2-\bar{A})+(2/\bar{\lambda}_3^2)(1+1/\bar{\lambda})^3\bar{U}/\bar{T}. \tag{44}$$

Higher derivatives of A can be calculated by using eqn (43) and simplifications resulting from eqns (41) and (9). We note that

$$U' = \lambda_3^{-1}(1-1/\lambda^2)(W_{1111}+4\lambda_3^2W_{1112}+6\lambda_3^4W_{1122}+4\lambda_3^6W_{1222}+\lambda_3^8W_{2222}),$$

etc.

Using eqns (20), (21) and (44) we obtain

$$D_1 = \frac{4(2\bar{\lambda}+1)}{(\bar{\lambda}-1)^3}+\frac{\bar{A}}{\bar{\lambda}(\bar{\lambda}^2-1)}\left\{3\bar{\lambda}-1+\frac{1}{2}(\bar{\lambda}-1)^2\bar{A}\right\}-(2/\bar{\lambda}_3^2)(1+1/\bar{\lambda})^3\bar{U}/\bar{T}. \tag{45}$$

Whence, with eqn (9), $D_1 > 0$ provided

$$\bar{U} < \frac{2\bar{\lambda}_3^2\bar{\lambda}^3(2\bar{\lambda}+1)\bar{T}}{(\bar{\lambda}^2-1)^3}\left\{1+\frac{1}{8}\bar{A}\frac{(\bar{\lambda}-1)^3(\bar{\lambda}^2-2\bar{\lambda}+3)}{\bar{\lambda}(\bar{\lambda}+1)(2\bar{\lambda}+1)}\right\}. \tag{46}$$

Because of the central role played by the Mooney–Rivlin material in theoretical studies, it is desirable to obtain in more explicit form certain results for it that follow from those derived above. Thus, if W is a linear function of I_1 and I_2, then $A \equiv 0$ and $\bar{\lambda}$ is the positive root of (cf. eqn (9))

$$\bar{\lambda}^3-3\bar{\lambda}^2-\bar{\lambda}-1 = 0 \tag{47}$$

i.e.

$$\bar{\lambda} = 1+2^{1/3}[(1+\surd(11/27))^{1/3}+(1-\surd(11/27))^{1/3}] \simeq 3.3829\ldots.$$

From eqns (5) we find $Q = \lambda+1$, $R = \lambda-1$, so that

$$\Delta = \bar{Q}-\bar{R} = 2, \qquad \bar{Q}+\bar{R} = 2\bar{\lambda} \tag{48}$$

and, from eqns (15)

$$\begin{aligned} &r_1 = 2/(\bar{\lambda}-1), \qquad r_2 = 2/(\bar{\lambda}-1)^2, \qquad q_1 = 2(\bar{\lambda}-1)/(\bar{\lambda}+1)^2 \\ &q_2 = 2/(\bar{\lambda}+1)^2, \qquad q_j = r_j = 0, \qquad j \geqslant 3. \end{aligned} \tag{49}$$

We may employ eqn (47) to express $\bar{\lambda}^3$ in terms of lower-degree powers of $\bar{\lambda}$. The result of applying this procedure in eqns (20), (21) and (23) yields

$$d_0 = 2\bar{\lambda}/D_1, \qquad D_1 = 2+1/\bar{\lambda}. \tag{50}$$

Reductions of this type will be routinely made in what follows without further mention.

The substitution from eqns (48) and (49) into eqns (17), (19) and $(30)_2$ yields

$$\begin{aligned} &R_1 = (\bar{\lambda}-1)^{-1}, \quad Q_1 = (\bar{\lambda}+1)^{-1}; \qquad R_j = Q_j = 0, \quad j=2,3; \qquad E_j = 0, \qquad j = 1,2,3; \\ &N_1 = \tfrac{1}{2}(3\bar{\lambda}^2+6\bar{\lambda}-1)/(3\bar{\lambda}^2+1), \quad N_2 = \tfrac{3}{2}(\bar{\lambda}+1)/(3\bar{\lambda}^2+1), \quad N_3 = 1/\{2(3\bar{\lambda}^2+1)\}, \\ &D_2 = -\tfrac{3}{4}(\bar{\lambda}-1)^2/(2\bar{\lambda}+1), \quad D_3 = -\tfrac{1}{4}(\bar{\lambda}-1)/(2\bar{\lambda}+1), \quad D_4 = 0 \end{aligned} \tag{51}$$

and we employ eqns (24) to obtain

$$\begin{aligned} \delta_1 &= 2(2\bar{\lambda}^2+\bar{\lambda}+1)/\{(3\bar{\lambda}^2+1)(2\bar{\lambda}+1)(\bar{\lambda}+1)(\bar{\lambda}-1)\} \\ \delta_2 &= 4(35\bar{\lambda}^2+12\bar{\lambda}+9)/\{(3\bar{\lambda}^2+1)(2\bar{\lambda}+1)^2(\bar{\lambda}+1)(\bar{\lambda}-1)^2\} \\ \delta_3 &= -4(270\bar{\lambda}^2+100\bar{\lambda}+79)/\{(3\bar{\lambda}^2+1)(2\bar{\lambda}+1)^3(\bar{\lambda}+1)(\bar{\lambda}-1)^3\}. \end{aligned} \tag{52}$$

Upon noting that $E_j = 0, j = 1,2,3$, the use of eqns (39) and (40), with eqn (48), yields, for the Mooney–Rivlin material

$$\begin{aligned} \lambda = \bar{\lambda}\{1 - vd_0\,\mathrm{e}^{-2\eta} + \bar{\lambda}d_0^2\delta_1\mathrm{e}^{-4\eta} - v\bar{\lambda}^2d_0^3(\delta_1^2+\delta_2)\,\mathrm{e}^{-6\eta} + vd_0\,\mathrm{e}^{-2\bar{\lambda}\eta} \\ + \bar{\lambda}^3d_0^4(\delta_1^3+3\delta_1\delta_2+\delta_3)\,\mathrm{e}^{-8\eta} + \langle -2(\bar{\lambda}+1)\eta\rangle\} \end{aligned} \tag{53}$$

where the δ's are given by eqns (52), and from eqns (50)

$$d_0 = 2\bar{\lambda}^2/(2\bar{\lambda}+1). \tag{54}$$

Series (53) could also be developed by using the Mooney–Rivlin form of *W ab initio*. The relevant bifurcation equation from eqn (4) is found to be

$$\frac{\sinh(\lambda+1)\eta}{\sinh(\lambda-1)\eta} = v\frac{\lambda+1}{\lambda-1}\frac{\{(\lambda-1)^2+\lambda(\lambda+1)^2\}}{\{(\lambda+1)^2-\lambda(\lambda-1)^2\}} \tag{55}$$

which is equivalent to the alternate version[2,10]

$$\tanh \lambda\eta/\tanh \eta = \{4\lambda^3/(\lambda^2+1)^2\}^v. \tag{56}$$

Comparison of the resulting algebraic forms of the coefficients with those in series (53) can be aided upon noting the following identities, all of which are equivalent to eqn (47):

$$\begin{aligned} 4\bar{\lambda}^3 = (\bar{\lambda}^2+1)^2, \qquad (\bar{\lambda}+1)^2 = \bar{\lambda}(\bar{\lambda}-1)^2, \qquad (\bar{\lambda}-1)^3 = 4\bar{\lambda}, \\ 3\bar{\lambda}^2+1 = \bar{\lambda}(\bar{\lambda}+1)(\bar{\lambda}-1), \qquad \bar{\lambda}-3 = (\bar{\lambda}+1)/\bar{\lambda}^2. \end{aligned} \tag{57}$$

It has been noted previously[4–6] that eqn (47) arises as *the* condition defining the critical deformation for a "half-space" of Mooney–Rivlin (or neo-Hookean) material.

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